Introduction

Calculating the derivative of a function $f(x)$ can sometimes be a very tedious task when you have to spend time applying the formula for the derivative $$f'(x)=\lim_{a \to 0} \frac{f(x+a)-f(x)}{a}.$$ In this article, we present three rules to make calculating the derivative, a very simple process! These methods will help you with your exams! The rules can be used to make differentiating some functions very easy. So you can have a full understanding, I have presented a proof of each trick so you will understand why it works.

Sum Rule

One of the most common and most useful tricks to calculate derivatives is the sum rule. The sum rule states that if $f(x)=g(x)+h(x),$ then $f'(x)=g'(x)+h'(x).$ In other words, the sum rule states that if one function is the sum of two other functions, then the derivative of that function is simply the sum of the derivatives of the two other functions.

Now, let’s look at the proof of this.

First, we have $$f'(x)=\lim_{a \to 0} \frac{f(x+a)-f(x)}{a}.$$ Since $f(x)=g(x)+h(x),$ we can rewrite this as $$f'(x)=\lim_{a \to 0} \frac{(g(x+a) + h(x+a))-(g(x)+h(x))}{a}.$$ Arranging terms in the numerator yields $$f'(x)=\lim_{a \to 0}\frac{(g(x+a)-g(x)) + (h(x+a)-h(x))}{a}.$$ We can then rewrite this as $$f'(x)=\lim_{a \to 0} \frac{g(x+a)-g(x)}{a}+\lim_{a \to 0} \frac{h(x+a)-h(x)}{a}=g'(x)=h'(x)$$ thus proving the sum rule.

Let’s try a couple examples now.

Example 1: Calculate the derivative of $f(x)=x^2+3x.$

Solution: The function f is the sum of x^2 and $3x.$ The derivative of x^2 is $2x$ and the derivative of $3x$ is $3.$ So, the derivative of x^2+3x is $2x+3.$

Example 2: Calculate the derivative of $f(x)=x^2+\frac{1}{x}+2x+9.$ The derivative of x^2 is $2x,$ the derivative of $\frac{1}{x}$ is $-\frac{1}{x^2},$ the derivative of $2x$ is $2,$ and the derivative of 9 is $0,$ so the derivative of f is $2x-\frac{1}{x^2}+2.$

Constant Multiple Rule

Another rule that makes calculating derivatives easier is the constant multiple rule. The constant multiple rule states that when $f(x)=k \cdot g(x),$ for some constant $k,$ then $f'(x)= k \cdot g'(x).$ In other words, this rule says that when a function is a constant multiplied by another function, the derivative of the function is the constant multiplied by the derivative of the other function.

Proof:

We start with $$f'(x)=\lim_{a \to 0} \frac{f(x+a)-f(x)}{a}.$$ Using the fact that $f(x)=k \cdot g(x),$ we get $$f'(x)=\lim_{a \to 0} \frac{k \cdot g(x+a)-k \cdot g(x)}{a}=k \cdot \lim_{a \to 0} \frac{g(x+a)-g(x)}{a}=k \cdot g'(x)$$ which completes the proof.

Example: What is the derivative of $f(x)=5x^2+\frac{3}{x}?$

Solution: From the sum rule, we need to find the derivatives of $5x^2$ and $\frac{3}{x}$ individually and then add them. By the product rule, the derivative of $5x^2$ is $5 \cdot 2x=10x$ and the derivative of $\frac{3}{x}$ is $3 \cdot -\frac{1}{x^2}=\frac{-3}{x^2}.$ Thus, $f'(x)=10x-\frac{3}{x^2}.$

Power Rule

The final rule we will look at is the power rule. The power rule states that if $f(x)=x^n,$ then $f'(x)=n \cdot x^{n-1}.$ As the rule is very straightforward, there is no need for a written explanation of it.

Proof:

This rule holds for any $n \in \mathbb{R},$ but to prove that we would have to use implicit differentiation and logarithmic differentiation which you might not know. It is a much easier task to prove the rule for $n \in \mathbb{Z}^+$ which is what we will do now. We start by using the alternative formula for the derivative. $$f'(x)=\lim_{a \to x} \frac{f(a)-f(x)}{a-x}.$$ We can use our definition $f(x)=x^n$ and factor the numerator to get

$$\begin{aligned}
f'(x) &= \lim\nolimits{a \to x} \frac{a^n-x^n}{a-x} \\
&= \lim\nolimits{a \to x} \frac{(a-x)(a^{n-1}+xa^{n-2} + \dots + x^{n-2}a + x^{n-1})}{a-x} \\
&= \lim\nolimits{a \to x} (a^{n-1}+xa^{n-2} + \dots + x^{n-2}a + x^{n-1})
\end{aligned}$$

As a approaches $x,$ we have

$$\begin{aligned}
f'(x) &= x^{n-1}+xx^{n-2}+\dots + x^{n-2}x + x^{n-1} \\
&= x^{n-1}+\dots+x^{n-1} \\
&= n \cdot x^{n-1}
\end{aligned}$$

which completes the proof.

Although we only proved this rule for the positive integers, we can apply this for any real number so we will use the rule for all real numbers in the examples.

Example 1: Find the derivative of $x^3.$

Solution: We apply the power rule to get $3x^2.$

Example 2: Find the derivative of $\frac{1}{x}.$

Solution: We can first rewrite $\frac{1}{x}$ as $x^{-1}.$ Applying the power rule gives $-1 \cdot x^{-2} = -1 \cdot \frac{1}{x^2}=-\frac{1}{x^2}$ as the derivative.

Example 3: Find the derivative of $\sqrt{x}.$

Solution: We can rewrite $\sqrt{x}$ as $x^{1/2}.$ We apply the power rule to get $\frac{1}{2} \cdot x^{-1/2}$ as the derivative. Simplifying gives $\frac{1}{2\sqrt{x}}=\frac{\sqrt{x}}{2x}$ as the derivative.